The Use of Absorbing Boundary Conditions in the Method of Lines

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***Abstract*—An artificial lossy absorbing boundary condition is proposed for use in the method of lines for simulating unbounded electromagnetic structures. It is found, through the modeling results of a microstrip line, that the proposed absorbing boundary condition could be effective and a simple rule of application may be established.**

I. Introduction

The proposal of an absorbing boundary condition (ABC) [1] for the method of lines (MoL) paves the way for effective simulation and modeling of unbounded electromagnetic structures. The ABC is derived from a factorization technique of Helmholtz wave operator. Very recently, the perfectly-matched layer (PML) technique [2] has stimulated a great interest in both time-domain and frequency-domain simulation of electromagnetic scattering, radiation, and propagation problems. The PML technique involves the application of a nonphysical absorbing material adjacent to the computational boundary. The PML material has characteristics that permit electromagnetic waves of arbitrary frequency and angle of incidence to be absorbed while maintaining the impedance and velocity of a lossless dielectric [3]. This technique has shown its effectiveness and generality of handling a large class of problems in discrete domain techniques.

Inspired by the PML concept, this work attempts to introduce an artificial lossy factor in the ABC that was formulated for the method of lines [1]. Results show that effectiveness of the ABC may be reinforced and the computational window required to achieve the same accuracy of result may become smaller.

II. Formulation

Considering a simple example of searching for the effective permittivity $\varepsilon_{\mathrm{re}}$ of a unbounded microstrip line, it is known that the ABC used in the MoL is derived from the wave factorization at a specific boundary along the transverse direction, where the relative dielectric permittivity $\varepsilon_r(=\varepsilon_r')$ is real for a lossless case. The operator of the Helmholtz equation governing electric and magnetic potentials can be factorized into inbound and outbound parts as in [1]

$$L\psi = L^+L^-\psi = 0 \tag{1}$$

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with

$$L^{\pm} = D_{\bar{x}} \pm j\sqrt{\varepsilon_d}\sqrt{1+S^2}, \quad S^2 = D_{\bar{y}}^2/\varepsilon_d, \quad \varepsilon_d = \varepsilon_r - \varepsilon_{\mathrm{re}}. \tag{2}$$

The detail of (2) can be referred to [1]. It should be pointed out that the second-order approximation of $\sqrt{1+S^2} \approx p_0 + p_2S^2$ made in [1] is valid if and only if the absolute value of S is smaller than one. Although it is true that p_0 and p_2 can be chosen to be a set of specific values according to the Taylor series at a designated point of S, and it also true that the use of the much better Padé approximation may lead to some additional improvement, it is our interest to pursue the same approximation as in [1] for the purpose of comparison. In the above second-order approximation, p_0 and p_2 are chosen to be 1.0 and 0.5, respectively. The wave factorization (complex term) and Taylor approximation will inevitably introduce a complex propagation constant for the lossless case. It is obvious that the absolute value of S depends analytically on ε_d as long as the discretization scheme and size of computational window (a) are fixed. The differential operator $D_{\bar{y}}^2$ seems somewhat to be stationary. Looking into the explicit formulation of $D_{\bar{y}}^2$ suggests that

$$|S| = \left|\frac{D_{\bar{y}}}{\sqrt{\varepsilon_d}}\right| = \left|\sqrt{\frac{\delta^{xx}/(h_x^2k_0^2)}{\varepsilon_d} - 1}\right| \tag{3}$$

in which δ^{xx} stands for eigenvalues of the second-order x-directed finite-difference operator. This equation indicates that choosing a fine discretization size h_x cannot guarantee automatically $|S| \leq 1.0$ unless a undesirable large size of the computational window (a) is selected. To keep the value of (a) to be minimized, there are two possibilities: selecting alternative set of value for p_0 and p_2, or introducing an artificial lossy parameter added in the term of dielectric permittivity at the boundary of interest. The first technique [4] has to deal with two parameters while the second strategy is relatively simple. In the lossless case, the approximation will be certainly reinforced by applying the second condition such that $|S| \leq 1.0$ is always satisfied. This is done by defining a nonphysical lossy layer with a thickness of h_x and complex term $\varepsilon_d = \varepsilon_r - \varepsilon_{\mathrm{re}}$ with $\varepsilon_r = \varepsilon_r' - j\varepsilon_r''$ for the outer boundaries. In this way, a better absorbing condition may be obtained by choosing an appropriate value of ε_r''.

III. Results and Discussion

Naturally, the use of the artificial lossy boundary condition also leads to a complex propagation constant of which the

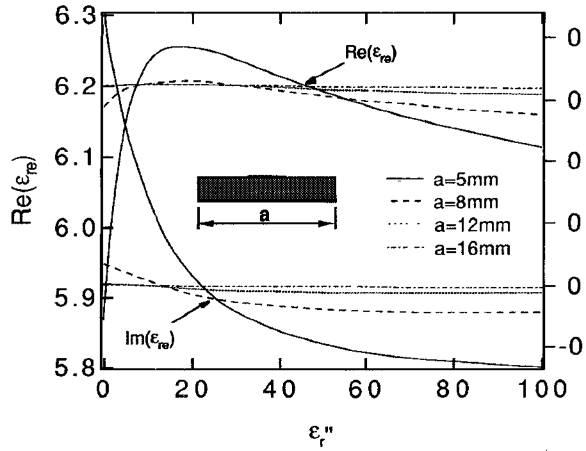

Fig. 1. Convergence feature of complex effective permittivity of a unbounded microstrip line with respect to the value of the proposed artificial lossy factor for different computational windows (strip width $w = 1$ mm, thickness of dielectric substrate $d = 1$ mm, $f = 5$ GHz, $\varepsilon_r' = 8.875$, and nine lines intersecting the strip).

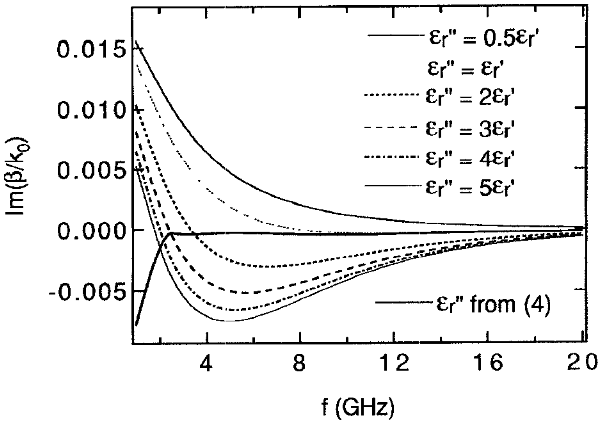

Fig. 2. Electrical performance of different artificial lossy factor ε_r'' and the proposed empirical formula on the imaginary part of the normalized propagation constant as a function of frequency with the computational window of $a = 8$ mm (the same geometrical and electrical conditions as in Fig. 1).

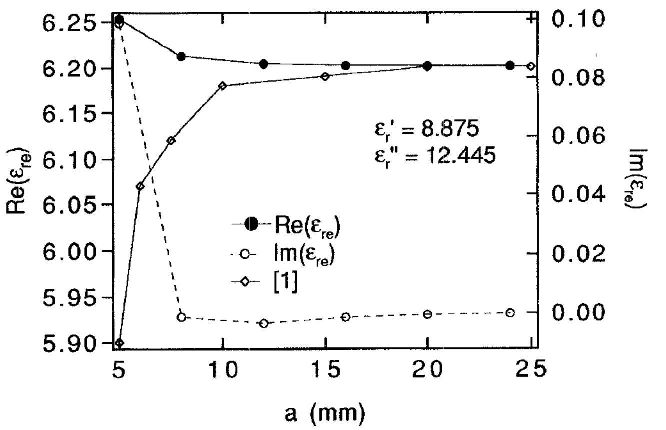

Fig. 3. Characteristics of complex effective permittivity versus the size of computational window with the artificial lossy factor obtained by the empirical formula compared to [1] (the same geometrical and electrical conditions as in Fig. 1).

imaginary part can, however, be minimized into a negligible level. This can be achieved by choosing certain value of ε_r''.

Fig. 1 shows a monotonic concurrent convergence of complex effective permittivity as the artificial lossy factor increases for different size of computational window. The results suggest that the original implementation of the ABC with $\varepsilon_r'' = 0$ leads to a positive imaginary part of the effective permittivity that is not physical (see the imaginary part). Clearly, the introduction of nonzero ε_r'' leads to better results in most situations, in particular for smaller computational windows. Nevertheless, the computational window should have a relative width "a/w" greater than or at least equal to eight. In this case, the effective permittivity is quite stable with regard to the choice of the artificial lossy factor.

Fig. 2 presents the imaginary part of the normalized propagation constant as a function of frequency with different artificial lossy factor ε_r'' ranging from low to high value with respect to ε_r'. It should be noted that the real part of the normalized propagation constant that is not shown in the letter for brevity remains almost the same regardless of the high value of ε_r'' as long as the condition of $\varepsilon_r'' \geq \varepsilon_r'$ is satisfied or a large computational window "a" is selected. The imaginary part that is not shown in [1] without considering the artificial lossy factor may be positive. Under the proposed artificial lossy condition, the nonphysical imaginary part of the propagation constant for a lossless structure can be significantly decreased as shown in Fig. 2 for a large value of ε_r''. It seems that the optimum choice of ε_r'' is equal to (the limiting case) or higher than ε_r' in our calculations. More precisely, the error range in the case of $\varepsilon_r'' \geq \varepsilon_r'$ will be reduced at least by half compared to the results without using the artificial lossy factor. Considering the limiting case of the artificial lossy factor $\varepsilon_r'' = \varepsilon_r'$ that yields an imaginary value in the proximity of zero over a large frequency band of interest, the optimum value of ε_r'' should be chosen such that the imaginary part is reduced to a negligible negative value. As indicted in Figs. 1 and 2, such an empirical relation $\varepsilon_r'' \geq \varepsilon_r'$ is found to be not so sensitive to higher frequency and line parameters of guiding structure as long as the results at lower frequency become satisfactory owing to the field confinement at higher frequency. This is attributed to the fact that the artificial loss factor is actually used in reducing the effect of the Taylor approximation on the propagation constant. Under the condition of "a/w" ≥ 8, an empirical formula for selecting the artificial lossy factor is proposed in the following:

$$\varepsilon_r'' = \varepsilon_r' + \frac{\varepsilon_r'}{0.030f^3 - 0.149f^2 + 0.587f - 0.458} \tag{4}$$

in which the unity of frequency is GHz and $f \neq 0$. As shown in Fig. 2, the imaginary part obtained by using this formula is always negative, but very close to zero, for frequency greater than 1 GHz. Fig. 3 shows clearly the advantages of adding the artificial lossy factor which yields much more stable and accurate results for a smaller computational window (a) compared to the case without using ε_r'', and at the same time the imaginary part of $\varepsilon_{\rm re}$ tends to be zero if "a/w"

≥ 8 and the proposed empirical formula (4) is satisfied. This demonstrates a useful feature of the lossy absorbing boundary for high numerical efficiency. This is in particular of interest for three-dimensional unbounded problems.

IV. CONCLUSION

A lossy ABC is proposed for effective use of the method of lines for unbounded electromagnetic problems. Results indicate that the added lossy factor makes it possible to choose a smaller computational window with a better numerical accuracy compared to the conventional technique. The artificial lossy factor can be chosen by applying the empirical formula. It may come to conclude that the physical approximation of an absorbing boundary condition through truncation could be "compensated" by using an appropriate artificial (nonphysical) model.

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